Examples from understanding real numbers and sets and notation.

Question:

① Is $3 \in \mathbb{R}$?

Meaning: "Is 3 an element of the set of real #s?"

"Is 3 a real #?"

Answer: yes

② Is $\pi^e \in \mathbb{Q}$?

Meaning: "Is $\pi^e$ an element of the set of rational #s?"

"Is $\pi^e$ rational?"

"Does there exist two integers $a, b$ (with nonzero) s.t. $\pi^e = \frac{a}{b}$?" math meaning

Answer: No (we think)

③ Is it true that for some $x \in \mathbb{R}$, $\forall \varepsilon > 0$, $x < \varepsilon$?

Math meaning: True or False: $\exists x \in \mathbb{R}$ s.t. $\forall \varepsilon > 0, x < \varepsilon$.

Important: This is a legit math statement – all variables are quantified.

"Is there a real # x that is less than all positive real #s?"

"Is there a real # x that is $\leq 0$?"

<u>Answer</u>: Yes

<u>Proof</u>: Let $x = 0 \in \mathbb{R}$. Then $x \leq \varepsilon$ $\forall \varepsilon > 0$.

$\therefore$ x satisfies the condition, so the statement is true $\blacksquare$

④ Negation of the last one.

<u>True or False</u>: $\neg(\exists x \in \mathbb{R}$ s.t. $\forall \varepsilon > 0, x < \varepsilon)$

$\equiv \forall x \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $\neg(x < \varepsilon)$

$\equiv \forall x \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $x \geq \varepsilon$

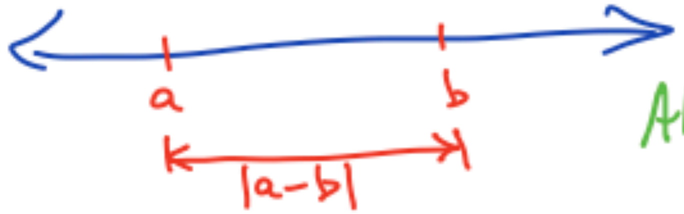
<u>Ans</u>: This is false.

<u>Proof</u>: Let $x = -1$. Then $\forall \varepsilon > 0$, it is not true that $x \geq \varepsilon$.

Thus the statement is false. $\blacksquare$

(Proof by counterexample.)

☆ If $a, b \in \mathbb{R}$, what is $|a-b|$?




Also, note $(a-b) = -(b-a)$
$\Rightarrow |a-b| = |b-a|$

☆ Another important result: triangle inequality:

- If $A, B \in \mathbb{R}$, then
$$|A+B| \leq |A| + |B|.$$
- If $C, D \in \mathbb{R}$, then
$$|C-D| \geq |C| - |D|$$
also $$|C-D| \geq |D| - |C|$$
so in fact
$$|C-D| \geq \big||C| - |D|\big|$$

usually don't use, but it's true.

in $\mathbb{R}^2$


$|A+B| \leq |A| + |B|$
triangle fact.

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<u>Example:</u> Prove that $\forall a, b \in \mathbb{R}$ s.t. $a < b$, $\exists c \in \mathbb{R}$ s.t. $a < c < b$.

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$\boxed{\text{Pf.}}$ Let $c = \frac{a+b}{2}$. Observe that
$\frac{a+b}{2} > \frac{a+a}{2} = a$. ✓ Also observe that

$\frac{a+b}{2} < \frac{b+b}{2} = b$. ✓ Thus, $a < c < b$. ∎